

NORMAL COVARIANT QUANTIZATION MAPS

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ABSTRACT. We consider questions related to quantizing complex valued functions defined on a locally compact topological group. In the case of bounded functions, we generalize R. Werner's approach to prove the characterization of the associated normal covariant quantization maps.

1. INTRODUCTION

Quantization is a procedure which associates a quantum mechanical observable to a given classical dynamical variable, the latter being represented by a complex valued Borel function on the phase space X of a classical system. The phase space X can be taken to be \mathbb{R}^{2n} or, more generally, G/H , where G is a locally compact second countable topological group and H a closed subgroup. We consider here only the case where $X = G$. Quantization can be realized e.g. by integrating the classical variable f with respect to a suitable (positive normalized) operator measure $E : \mathcal{B}(G) \rightarrow L(\mathcal{H})$, where $\mathcal{B}(G)$ is the Borel σ -algebra of subsets of G and $L(\mathcal{H})$ the set of bounded operators acting on the Hilbert space \mathcal{H} of the quantum system. The resulting operator integral $L(f, E) = \int f dE$ is a (possibly unbounded) linear operator, which is symmetric if f is real valued. (See Section 6 for our definition of the domain of the operator integral.) In many cases, the operator $L(f, E)$ is essentially selfadjoint, so that it is eligible to represent a quantum observable. The map $f \mapsto L(f, E)$ is linear (in the sense made precise in Section 6). If f is bounded, then $L(f, E) \in L(\mathcal{H})$. The operator integral has a convergence property, which could be called "quasicontinuity" (see e.g. [2, p. 22]): If (f_n) is an increasing sequence of positive Borel functions converging pointwise to a Borel function f , and $\varphi \in \mathcal{H}$ is a vector belonging to the domains of $L(f, E)$ and each $L(f_n, E)$, then the sequence $(\langle \psi | L(f_n, E) \varphi \rangle)$ of numbers converges for each $\psi \in \mathcal{H}$ to $\langle \psi | L(f, E) \varphi \rangle$.

As noted above, quantization might be any mapping Γ from the set of Borel functions to the set of linear operators on \mathcal{H} . It is therefore natural to ask which of them can be represented by operator integrals with respect to some positive operator measures. Essential requirements for Γ are linearity, positivity, the property that bounded functions are mapped to $L(\mathcal{H})$, and quasicontinuity, as they assure that the association $B \mapsto \Gamma(\chi_B)$ defines a positive operator measure E^Γ . Obviously, this does not guarantee that the quantization map Γ would coincide with the map given by the operator integral with respect to E^Γ ; in particular, nothing has been said about the domains of the operators $\Gamma(f)$. In the case of a bounded function f , however, the domain of the operator integral $L(f, E^\Gamma)$ is all of \mathcal{H} , and it follows easily that $\Gamma(f) = L(f, E^\Gamma)$. Thus, if we have a positive and quasicontinuous linear quantization map Γ , which maps bounded functions to $L(\mathcal{H})$, then (at least) the restriction of Γ to the set of bounded functions can be represented as the operator integral $L(\cdot, E^\Gamma)$.

Since the phase space G has a left Haar measure λ , it is convenient to consider the functions in $L^\infty(G, \lambda)$ (i.e. λ -equivalence classes of λ -essentially bounded λ -measurable complex functions) instead of bounded Borel functions. Assume that the original quantization map Γ (defined

on all complex Borel functions) is linear, positive, has the quasicontinuity property, and maps bounded functions to $L(\mathcal{H})$. In addition, we can require that each complex measure $B \mapsto E_{\psi,\varphi}^\Gamma(B) = \langle \psi | E^\Gamma(B) \varphi \rangle$ is λ -continuous. This ensures that $\Gamma(f)$ does not depend on the (Borel) representative of $f \in L^\infty(G, \lambda)$, so we get a well-defined positive linear quantization map $\tilde{\Gamma} : L^\infty(G, \lambda) \rightarrow L(\mathcal{H})$ which coincides with the map obtained from $L(\cdot, E^\Gamma)$ in the similar way.

When we restrict our attention to the positive linear quantization maps Γ defined on $L^\infty(G, \lambda)$ with values in $L(\mathcal{H})$, the condition of quasicontinuity is not appropriate, as it involves pointwise convergence. Instead, we require the somewhat similar condition of normality, i.e. weak-* continuity associated with the dualities $L^1(G, \lambda)^* = L^\infty(G, \lambda)$ and $\mathcal{T}(\mathcal{H})^* = L(\mathcal{H})$, where $L^1(G, \lambda)$ is the set of λ -equivalence classes of λ -integrable complex functions and $\mathcal{T}(\mathcal{H})$ is the set of all trace class operators on \mathcal{H} . Thus, if the λ -continuity of the complex measures $E_{\psi,\varphi}^\Gamma$ is assumed, we have $\Gamma = L(\cdot, E^\Gamma)$. Conversely, if a positive operator measure E is given, for which each $E_{\psi,\varphi}$ is λ -continuous, then the map $L^\infty(G, \lambda) \ni f \mapsto L(f, E) \in L(\mathcal{H})$ is linear, positive, and normal (see Section 5).

An important property of a quantization map $\Gamma : L^\infty(G, \lambda) \rightarrow L(\mathcal{H})$ (or the corresponding operator measure) is covariance (see Section 2), which connects it to the structure of the phase space. Covariance also conveniently implies the λ -continuity of the complex measures $E_{\psi,\varphi}^\Gamma$ (see section 5). Covariant positive phase space operator measures have proved highly useful also in various other applications of quantum mechanics, including for instance the fundamental questions on joint measurements of position and momentum observables and the problem of quantum state estimation (quantum tomography). Consequently, the structure of such operator measures has been studied extensively: the canonical examples of the covariant phase space observables are constructed e.g. in [5], whereas a complete group theoretical characterization is given in [3].

The characterization of [3] is based on a generalization of Mackey's imprimitivity theorem [4]. However, in the concrete case where the phase space is \mathbb{R}^{2n} , there is another, more direct (and completely different) approach, outlined by Holevo [10], and further elaborated by Werner [15]. In fact, in [15], Werner characterizes all the positive normal phase space translation covariant maps $\Gamma : L^\infty(\mathbb{R}^{2n}) \rightarrow L(\mathcal{H})$. The essential part of both Holevo's and Werner's proofs relies on the fact that the Banach space of trace class operators on a separable Hilbert space has the Radon-Nikodým property.

In this paper we generalize Werner's approach to the case where the phase space is a locally compact unimodular topological group, paying due attention to the details arising in this context. In addition, we consider briefly the question of quantization of unbounded functions.

2. PRELIMINARIES

If \mathcal{H} is a Hilbert space, we let $L(\mathcal{H})$ and $\mathcal{T}(\mathcal{H})$ denote the sets of bounded operators and trace class operators on \mathcal{H} , respectively.

Let μ_L denote the Lebesgue measure of \mathbb{R}^{2n} . Denote the Weyl operators on the Hilbert space $L^2(\mathbb{R}^n)$ by $W(x)$, $x = (q, p) \in \mathbb{R}^{2n}$, so that $W(q, p)$ acts according to

$$(W(q, p)\psi)(t) = e^{i\frac{1}{2}q \cdot p} e^{ipt} \psi(t + q).$$

They satisfy the relation

$$(1) \quad W(x)W(y) = e^{i\frac{1}{2}\{x,y\}} W(x+y),$$

where $\{(q, p), (q', p')\} = q \cdot p' - p \cdot q'$ for all $(q, p), (q', p') \in \mathbb{R}^{2n}$.

For each $x \in \mathbb{R}^{2n}$, define $\gamma(x) : \mathcal{T}(L^2(\mathbb{R}^n)) \rightarrow \mathcal{T}(L^2(\mathbb{R}^n))$ by $\gamma(x)(T) = W(x)TW(-x)$. Then the map $x \mapsto \gamma(x)$ has the following well-known properties. The proof is included for the reader's convenience.

- Lemma 1.** (a) $\gamma(x + y) = \gamma(x) \circ \gamma(y)$ for all $x, y \in \mathbb{R}^{2n}$.
(b) $\gamma(x)^*(A) = W(-x)AW(x)$ for all $A \in L(L^2(\mathbb{R}^n))$ and $x \in \mathbb{R}^{2n}$.
(c) $\gamma(x)$ is a positive trace-norm isometry for all $x \in \mathbb{R}^{2n}$.
(d) For each $A \in L(L^2(\mathbb{R}^n))$ and $S \in \mathcal{T}(L^2(\mathbb{R}^n))$, the function $x \mapsto \text{Tr}[A\gamma(x)(S)]$ is continuous.
(e) $\int \text{Tr}[P_1\gamma(x)(P_2)]d\mu_L(x) = (2\pi)^n$ for all one-dimensional projections P_1 and P_2 on $L^2(\mathbb{R}^n)$.

Proof. (a) is a direct consequence of the relation (1), and (b) follows from a basic property of the trace. If U is a unitary operator, $|USU^*| = U|S|U^*$ for each $S \in L(L^2(\mathbb{R}^n))$. Therefore, since $W(x)$ is unitary and $W(x)^* = W(-x)$, $\|\gamma(x)(S)\|_{\text{tr}} = \text{Tr}[|W(x)SW(-x)|] = \|S\|_{\text{tr}}$ for each $S \in \mathcal{T}(L^2(\mathbb{R}^n))$. This proves (c), as it is clear that $\gamma(x)$ is positive. To prove (d), take $A \in L(L^2(\mathbb{R}^n))$ and $S \in \mathcal{T}(L^2(\mathbb{R}^n))$. Let $x \in \mathbb{R}^{2n}$, and (x_n) be a sequence converging to x . Since $x \mapsto W(x)$ is strongly continuous, $\gamma(x_n)^*(A) = W(-x_n)AW(x_n) \rightarrow W(-x)AW(x) = \gamma(x)^*(A)$ weakly. Since all $W(x)$ are unitary, the sequence $(\gamma(x_n)^*(A))$ is norm bounded, from which it follows that it converges to $\gamma(x)^*(A)$ also ultraweakly. Thus we get

$$\text{Tr}[A\gamma(x_n)(S)] = \text{Tr}[\gamma(x_n)^*(A)S] \rightarrow \text{Tr}[\gamma(x)^*(A)S] = \text{Tr}[A\gamma(x)(S)],$$

which proves (d). The proof of (e) goes as follows. Assume that $P_1 = |\psi\rangle\langle\psi|$ and $P_2 = |\varphi\rangle\langle\varphi|$, where $\psi, \varphi \in \mathcal{H}$ are unit vectors. Define the function ϕ_q for each $q \in \mathbb{R}^n$ by $\phi_q(t) = \psi(t)\varphi(t+q)$. Then

$$1 = \|\psi\|^2\|\varphi\|^2 = \int \left(\int |\psi(t)|^2 |\varphi(q)|^2 dq \right) dt = \int \left(\int |\psi(t)|^2 |\varphi(t+q)|^2 dt \right) dq$$

by the Fubini-Tonelli theorem, so that $\phi_q \in \mathcal{H}$ for almost all q . By the unitarity of the inverse Fourier-Plancherel operator F , we have now

$$1 = \int \int |(F\phi_q)(p)|^2 dp dq.$$

But since ψ and $\varphi(\cdot + q)$ are in $L^2(\mathbb{R}^n)$, ϕ_q is also integrable, so

$$(F\phi_q)(p) = \frac{1}{\sqrt{(2\pi)^n}} \int e^{ip \cdot t} \phi_q(t) dt = \frac{1}{\sqrt{(2\pi)^n}} e^{-i\frac{1}{2}p \cdot q} \langle \psi | W(q, p) \varphi \rangle,$$

from which it follows that

$$(2\pi)^n = \int |\langle \psi | W(x) \varphi \rangle|^2 d\mu_L(x) = \int \text{Tr}[P_1\gamma(x)(P_2)]d\mu_L(x).$$

□

Now we proceed to a more abstract case.

If $(\Omega, \mathcal{A}, \nu)$ is a σ -finite (positive) measure space, we let $L^1(\Omega, \nu)$ denote the Banach space of (equivalence classes of) complex valued, ν -integrable functions, and $L^\infty(\Omega, \nu)$ the Banach space of (equivalence classes of) complex valued, ν -measurable, ν -essentially bounded functions.

A function g defined on Ω and having values in some Banach space is said to be ν -measurable, if for each $B \in \mathcal{A}$ of finite measure there is a sequence of ν -simple functions converging to $\chi_B g$ in

ν -measure (or, equivalently, there is a sequence of ν -simple functions which converges ν -almost everywhere to $\chi_B g$) [8, pp. 106, 150]. In the case where the value space of g is separable (in particular, if g is scalar-valued), ν -measurability is equivalent to the measurability with respect to the Lebesgue extension of the σ -algebra \mathcal{A} with respect to ν [8, p. 148]. If X is a Banach space, $\text{Iso}(X)$ denotes the group of linear homeomorphisms from X onto X .

Let \mathcal{H} be a separable Hilbert space. Let $\text{Aut}(\mathcal{T}(\mathcal{H}))$ denote the subgroup of $\text{Iso}(\mathcal{T}(\mathcal{H}))$ consisting of the positive maps which preserve the trace norm. The set $\text{Aut}(\mathcal{T}(\mathcal{H}))$ is equipped with the weak topology given by the set of functionals $u \mapsto \text{Tr}[Au(T)]$, where $A \in L(\mathcal{H})$, $T \in \mathcal{T}(\mathcal{H})$. For $u \in \text{Aut}(\mathcal{T}(\mathcal{H}))$, the adjoint map $u^* : L(\mathcal{H}) \rightarrow L(\mathcal{H})$ restricted to $\mathcal{T}(\mathcal{H})$ is equal to u^{-1} . It follows from the Wigner theorem that for each $u \in \text{Aut}(\mathcal{T}(\mathcal{H}))$ there is an either unitary or antiunitary operator U , such that $u(T) = UTU^*$ for all $T \in \mathcal{T}(\mathcal{H})$.

Let G be a locally compact unimodular second countable (Hausdorff) topological group, with Haar measure λ , such that there is a continuous group homomorphism $\beta : G \rightarrow \text{Aut}(\mathcal{T}(\mathcal{H}))$ and a constant $d > 0$, satisfying

$$(2) \quad \int \text{Tr}[P_1 \beta(g)(P_2)] d\lambda(g) = d$$

for all one-dimensional projections P_1 and P_2 on \mathcal{H} . The system (G, β, d) will remain fixed throughout the paper.

Remark.

- (a) It follows from Lemma 1 that the additive group \mathbb{R}^{2n} , with the homomorphism γ and the constant $(2\pi)^n$ constitute an example of the abstract system (G, β, d) .
- (b) The fact that each $\beta(g)$ has the form $\beta(g)(T) = U(g)TU(g)^*$ for some unitary or antiunitary operator $U(g)$ implies that, in the case where G is connected, the map $g \mapsto U(g)$ is a projective unitary representation of G which satisfies the square integrability condition

$$\int |\langle \psi | U(g) \varphi \rangle|^2 d\lambda(g) = d$$

for all unit vectors $\psi, \varphi \in \mathcal{H}$. The theory of such representations and the associated covariant operator measures is well developed, see e.g. [1]. It can be noted that in the case of a nonunimodular group, the square integrability condition is no longer of the above form for some fixed d [7].

In this paper, however, we do not need the explicit structure of the map β given by the projective representation $g \mapsto U(g)$. Thus we will use only the abstract definition, with the map γ associated with the group \mathbb{R}^{2n} as a concrete example.

If S is a positive trace class operator and A a bounded positive operator, the function $G \ni g \mapsto \text{Tr}[A\beta(g)(S)]$ is positive. Concerning the integrability of such a function, the following lemma holds (with the understanding that $\infty \cdot 0 = 0$):

Lemma 2. *Let $S \in \mathcal{T}(\mathcal{H})$ and $A \in L(\mathcal{H})$ be positive operators. Then*

$$d^{-1} \int \text{Tr}[A\beta(g)(S)] d\lambda(g) = \text{Tr}[A] \text{Tr}[S].$$

In particular, if $S \neq O$, the function $g \mapsto \text{Tr}[A\beta(g)(S)]$ is integrable if and only if $A \in \mathcal{T}(\mathcal{H})$.

Proof. The proof consists of several stages.

- 1) Assume that A and S are one-dimensional projections. Since now $\text{Tr}[A] \text{Tr}[S] = 1$, it follows directly from the relation (2) that $d^{-1} \int \text{Tr}[A\beta(g)(S)]d\lambda(g) = \text{Tr}[A] \text{Tr}[S]$.
- 2) Assume that S is a positive nonzero trace class operator and A a one-dimensional projection. Then $S = \sum_{i=1}^{\infty} w_i |\varphi_i\rangle\langle\varphi_i|$, where (φ_i) is an orthonormal sequence in \mathcal{H} , the series converging in the trace norm, and $w_i \geq 0$, $\sum_i w_i = \text{Tr}[S]$. Since the map $T \mapsto \text{Tr}[A\beta(g)(T)]$ is linear and trace-norm continuous, we have $\text{Tr}[A\beta(g)(S)] = \sum_i w_i \text{Tr}[A\beta(g)(|\varphi_i\rangle\langle\varphi_i|)]$ for all g . Now the result 1) and the monotone convergence theorem give

$$\begin{aligned} \text{Tr}[A] \text{Tr}[S] &= \sum_i w_i \text{Tr}[A] \text{Tr}[|\varphi_i\rangle\langle\varphi_i|] = \sum_i w_i d^{-1} \int \text{Tr}[A\beta(g)(|\varphi_i\rangle\langle\varphi_i|)]d\lambda(g) \\ &= d^{-1} \int \text{Tr}[A\beta(g)(S)]d\lambda(g). \end{aligned}$$

- 3) Let S be as before, and A a bounded positive operator such that the set $\sigma_p(A)$ of eigenvalues of A equals either the spectrum $\sigma(A)$ or the set $\sigma(A) \setminus \{0\}$. (In particular, all the positive compact operators are like this.) Now $E^A(\sigma_p(A)) = I$, where E^A is the spectral measure of A . It follows that the eigenvectors of A constitute an orthonormal basis of \mathcal{H} . Since \mathcal{H} is separable, the set $\sigma_p(A)$ is at most countable. Let $\sigma_p(A) = \{a_1, a_2, \dots\}$, and (φ_{nk}) be an orthonormal basis of \mathcal{H} , such that for each n , the vectors φ_{nk} span the eigenspace of A associated with the eigenvalue a_n . Now $\text{Tr}[A] = \sum_{nk} \langle \varphi_{nk} | A \varphi_{nk} \rangle = \sum_n a_n d_n$, where $d_n \leq \infty$ is the degree of the eigenvalue a_n . Moreover,

$$\text{Tr}[A\beta(g)(S)] = \sum_{nk} a_n \langle \varphi_{nk} | \beta(g)(S) \varphi_{nk} \rangle = \sum_{nk} a_n \text{Tr}[|\varphi_{nk}\rangle\langle\varphi_{nk}| \beta(g)(S)].$$

It now follows from 2) and the monotone convergence theorem that

$$d^{-1} \int \text{Tr}[A\beta(g)(S)]d\lambda(g) = \sum_n a_n d_n \text{Tr}[S] = \text{Tr}[A] \text{Tr}[S].$$

In particular, if A has an eigenspace of infinite dimension corresponding to a nonzero eigenvalue, then $\int \text{Tr}[A\beta(g)(S)]d\lambda(g) = \infty$.

- 4) Let again S be a positive nonzero trace class operator. Assume that A is a positive bounded operator, such that the set of eigenvalues of A equals neither the whole spectrum $\sigma(A)$ nor the set $\sigma(A) \setminus \{0\}$. Then $\sigma(A)$ contains a point $a_0 > 0$, which is not an eigenvalue of A . Now $E^A(\{a_0\}) = O$. It follows that $E^A(I_\epsilon)$, where $I_\epsilon = (a_0 - \epsilon, a_0 + \epsilon)$, is infinite-dimensional for all $\epsilon > 0$. Let $t = \frac{a_0}{2}$. Then $t\chi_{I_t}(x) \leq x$ for all $x \geq 0$, so that $tE^A(I_t) = \int t\chi_{I_t}(x)dE^A(x) \leq \int x dE^A(x) = A$. Since $E^A(I_t)$ is an infinite dimensional projection, $\infty = t\text{Tr}[E^A(I_t)] \leq \text{Tr}[A]$, and hence also $\text{Tr}[A] = \infty$. In addition, since $\beta(g)(S)$ is positive, $t\text{Tr}[E^A(I_t)\beta(g)(S)] \leq \text{Tr}[A\beta(g)(S)]$. Since the projection $E^A(I_t)$ is infinite-dimensional, 3) implies that the function $g \mapsto t\text{Tr}[E^A(I_t)\beta(g)(S)]$ is not integrable. Thus $d^{-1} \int \text{Tr}[A\beta(g)(S)]d\lambda(g) = \infty = \text{Tr}[A]\text{Tr}[S]$.

The lemma is proved. \square

In the following definition, note that the class of the function $f(g \cdot) \in L^\infty(G, \lambda)$ is independent of the representative of f .

Definition. A linear map $\Gamma : L^\infty(G, \lambda) \rightarrow L(\mathcal{H})$ is said to be β -covariant, if $\beta(g)^*(\Gamma(f)) = \Gamma(f(g \cdot))$ for all $f \in L^\infty(G, \lambda)$, $g \in G$.

The main result in this paper, Theorem 2, has the following rather straightforward and, at least in special cases, well-known converse.

Theorem 1. *Let T be a positive operator of trace one. Then for each $f \in L^\infty(G, \lambda)$, the integral*

$$(3) \quad d^{-1} \int f(g) \beta(g)(T) d\lambda(g)$$

exists as an operator $\Gamma(f) \in L(\mathcal{H})$ in the ultraweak sense. In addition, $\Gamma(g \mapsto 1) = I$, and the map $f \mapsto \Gamma(f)$ is linear, positive, normal, and β -covariant.

Proof. By Lemma 2, the function $g \mapsto \text{Tr}[S\beta(g)(T)]$ is in $L^1(G, \lambda)$ for each trace class operator S (the operator S can be written as a linear combination of four positive trace-class operators). Thus for each $f \in L^\infty(G, \lambda)$ we can define the (clearly linear) functional $\Phi_f : \mathcal{T}(\mathcal{H}) \rightarrow \mathbb{C}$ by

$$\Phi_f(S) = d^{-1} \int f(g) \text{Tr}[S\beta(g)(T)] d\lambda(g).$$

Let now $f \in L^\infty(G, \lambda)$ be real valued. If $S \in \mathcal{T}(\mathcal{H})$ is positive, we have by Lemma 2

$$|\Phi_f(S)| \leq d^{-1} M_f \int \text{Tr}[S\beta(g)(T)] d\lambda(g) = M_f \|S\|_{\text{tr}},$$

where $M_f < \infty$ is such that $f(g) \leq M_f$ for almost all g . If $S \in \mathcal{T}(\mathcal{H})$ is selfadjoint, it can be written in the form $S = S^+ - S^-$, where $S^\pm \in \mathcal{T}(\mathcal{H})$ are positive and $|S| = S^+ + S^-$. Now

$$|\Phi_f(S)| \leq |\Phi_f(S^+)| + |\Phi_f(S^-)| \leq M_f (\|S^+\|_{\text{tr}} + \|S^-\|_{\text{tr}}) = M_f \|S\|_{\text{tr}},$$

so that for real valued f , the map Φ_f restricted to the set of selfadjoint trace class operators is a real valued trace-norm continuous linear functional. Hence, there is a selfadjoint operator $\Gamma(f) \in L(\mathcal{H})$, such that $\Phi_f(S) = \text{Tr}[S\Gamma(f)]$ for all selfadjoint $S \in \mathcal{T}(\mathcal{H})$. For an arbitrary $S \in \mathcal{T}(\mathcal{H})$, we have $S = S_1 + iS_2$, where each $S_i \in \mathcal{T}(\mathcal{H})$ are selfadjoint, and so

$$\Phi_f(S) = \Phi_f(S_1) + i\Phi_f(S_2) = \text{Tr}[S_1\Gamma(f)] + i\text{Tr}[S_2\Gamma(f)] = \text{Tr}[S\Gamma(f)].$$

Let now $f \in L^\infty(G, \lambda)$ be complex valued: $f = f_1 + if_2$, where f_1 and f_2 are real. Then clearly $\Phi_f(S) = \Phi_{f_1}(S) + i\Phi_{f_2}(S) = \text{Tr}[S(\Gamma(f_1) + i\Gamma(f_2))]$ for all $S \in \mathcal{T}(\mathcal{H})$. Define $\Gamma(f) := \Gamma(f_1) + i\Gamma(f_2) \in L(\mathcal{H})$. Now

$$\Phi_f(S) = \text{Tr}[S\Gamma(f)]$$

for all $S \in \mathcal{T}(\mathcal{H})$ and $f \in L^\infty(G, \lambda)$, implying the existence of the integral (3) in the ultraweak sense as the operator $\Gamma(f) \in L(\mathcal{H})$.

The statement $\Gamma(g \mapsto 1) = I$ follows from Lemma 2.

Clearly $\Gamma : L^\infty(G, \lambda) \rightarrow L(\mathcal{H})$ is linear. If $f \geq 0$ and $\varphi \in \mathcal{H}$,

$$\langle \varphi | \Gamma(f) \varphi \rangle = \text{Tr}[\varphi \langle \varphi | \Gamma(f)] = \Phi_f(|\varphi\rangle \langle \varphi|) = d^{-1} \int f(g) \langle \varphi | \beta(g)(T) \varphi \rangle d\lambda(g) \geq 0,$$

which proves the positivity of Γ . Since $\Phi_f(S) = \text{Tr}[S\Gamma(f)]$ for all $S \in \mathcal{T}(\mathcal{H})$ and $f \in L^\infty(G, \lambda)$, Γ is the dual of the map $\mathcal{T}(\mathcal{H}) \ni S \mapsto d^{-1} \text{Tr}[S\beta(\cdot)(T)] \in L^1(G, \lambda)$, and hence normal.

Covariance is seen from the calculation

$$\begin{aligned}\mathrm{Tr}[S\Gamma(f(g\cdot))] &= d^{-1} \int f(gg') \mathrm{Tr}[S\beta(g')(T)] d\lambda(g') = d^{-1} \int f(g') \mathrm{Tr}[S\beta(g^{-1}g')(T)] d\lambda(g') \\ &= d^{-1} \int f(g') \mathrm{Tr}[\beta(g)(S)\beta(g')(T)] d\lambda(g') = \mathrm{Tr}[\beta(g)(S)\Gamma(f)] = \mathrm{Tr}[S\beta(g)^*(\Gamma(f))],\end{aligned}$$

where $g \in G$ and $f \in L^\infty(G, \lambda)$ are arbitrary, and the left invariance of λ , along with the properties of the map β , are used. \square

3. GENERAL COVARIANT MAPS

In this section we formulate the essential part of the characterization in yet a slightly more general context. Let X a Banach space having the Radon-Nikodým property, i.e., if $(\Omega, \mathcal{A}, \nu)$ is a finite (positive) measure space and $\mu : \mathcal{A} \rightarrow X$ a ν -continuous vector measure with bounded variation, there is a ν -(Bochner-)integrable function $g_\mu : \Omega \rightarrow X$, such that $\mu(B) = \int_B g_\mu d\nu$ for all $B \in \mathcal{A}$ (see [6, p. 61]). The function g_μ is ν -essentially unique [6, p. 47, Corollary 5].

The statement of the following Lemma is called the Riesz Representation Theorem. It is proved in [6, pp. 62-63], in the case where ν is a finite measure. The Lemma here is an obvious generalization of that result to the σ -finite case. As it constitutes the very starting point of the proof of the main result of the paper, we give its proof here.

Lemma 3. *Let $(\Omega, \mathcal{A}, \nu)$ be a σ -finite measure space, X a Banach space having the Radon-Nikodým property, and $\Gamma : L^1(\Omega, \nu) \rightarrow X$ a continuous linear map. Then there is a ν -essentially unique ν -measurable function $v : \Omega \rightarrow X$, such that $\sup_{x \in \Omega} \|v(x)\| = \|\Gamma\|$, and*

$$\Gamma(f) = \int f v d\nu$$

for all $f \in L^1(\Omega, \nu)$.

Proof. Choose a disjoint sequence (K_n) of sets in \mathcal{A} , such that $\Omega = \bigcup_n K_n$, and $\nu(K_n) < \infty$. Denote by ν_n the restriction of ν to the σ -algebra $\mathcal{A}(K_n) = \{B \cap K_n | B \in \mathcal{A}\}$. Define the set function $\mu_n : \mathcal{A}(K_n) \rightarrow X$ by $\mu_n(B) = \Gamma(\chi_B)$. Now $\|\mu_n(B)\| \leq \|\Gamma\| \|\chi_B\|_1 = \|\Gamma\| \nu_n(B)$ for all $B \in \mathcal{A}(K_n)$. It follows that μ_n is a ν_n -continuous vector measure of bounded variation, with the variation satisfying $|\mu_n|(B) \leq \|\Gamma\| \nu(B)$ for all $B \in \mathcal{A}(K_n)$. Since X has the Radon-Nikodým property and ν_n is a finite measure, there is a ν_n -integrable function $v_n : K_n \rightarrow X$, such that $\mu_n(B) = \int_B v_n d\nu_n$ for all $B \in \mathcal{A}(K_n)$. For each $f \in L^1(K_n, \nu_n)$, let \tilde{f} be the function $\Omega \rightarrow \mathbb{C}$ which coincides with f in K_n and is zero elsewhere. Since the map $L^1(K_n, \nu_n) \ni f \mapsto \Gamma(\tilde{f}) \in X$ is linear and continuous, it follows from [6, Lemma 4, p. 62] that $\|v_n(x)\| \leq \|\Gamma\|$ for ν_n -almost all $x \in K_n$, and $\Gamma(\tilde{f}) = \int f v_n d\nu_n$ for each $f \in L^1(K_n, \nu_n)$. By [6, Corollary 5, p. 47], v_n is ν_n -essentially unique, and v_n can be redefined to be zero in the null set in which originally $\|v_n(x)\| > \|\Gamma\|$. Now we have $\sup_{x \in K_n} \|v_n(x)\| \leq \|\Gamma\|$.

We denote by v_n also the function $\Omega \rightarrow X$ which coincides with v_n in K_n and is zero elsewhere. Since the sets K_n are disjoint, we can define $v = \sum_n v_n$, where the sum converges pointwise. Since v is a pointwise limit of ν -measurable functions, it is itself ν -measurable. Denote $M = \sup_{x \in \Omega} \|v(x)\| \leq \|\Gamma\|$.

Let $f \in L^1(\Omega, \nu)$. Now the sequence (f_k) , where $f_k = \chi_{\cup_{n=1}^k K_n} f$ converges pointwise, and hence (by the dominated convergence theorem) in $L^1(\Omega, \nu)$ to f . By continuity, $(\Gamma(f_k))$ converges to $\Gamma(f)$ in X . On the other hand, since $\|f_k(x)v(x)\| \leq |f(x)|\|\Gamma\|$ for all $x \in \Omega$, the

dominated convergence theorem gives

$$\Gamma(f_k) = \sum_{n=1}^k \Gamma(\chi_{K_n} f) = \sum_{n=1}^k \int (f|K_n) v_n d\nu_n = \int f_k v d\nu \longrightarrow \int f v d\nu.$$

Thus,

$$\Gamma(f) = \int f v d\nu.$$

Since $\|\Gamma(f)\| \leq \int |f(x)| \|v(x)\| d\nu(x) \leq \|f\|_1 M$ for all $f \in L^1(\Omega, \nu)$, we get $\|\Gamma\| \leq M$, so $M = \|\Gamma\|$. Since ν is σ -additive, v is ν -essentially unique by [6, Corollary 5, p. 47]. The Lemma is proved. \square

The next Proposition allows us to specify the nature of the function v obtained in the previous Lemma, in the case where Ω is a locally compact topological group possessing certain additional properties. The next Lemma is essential to its proof.

Lemma 4. *Let Ω be a locally compact second countable topological group with a left Haar measure ν .*

- (a) *Let $h : \Omega \rightarrow \mathbb{C}$ be a ν -measurable ν -essentially bounded function such that for each $y \in \Omega$, the function $h(y \cdot)$ coincides with h ν -almost everywhere. Then there is a constant $c \in \mathbb{C}$, such that $h(x) = c$ for ν -almost all $x \in \Omega$.*
- (b) *Let X be a Banach space, and $h : \Omega \rightarrow X$ a ν -measurable ν -essentially bounded function such that for each $y \in \Omega$, the function $h(y \cdot)$ coincides with h ν -almost everywhere. Then there is an $s \in X$, such that $h(x) = s$ for ν -almost all $x \in \Omega$.*

Proof. (a) Clearly the positive functions $h_i^\pm = \frac{1}{2}(|h_i| \pm h_i)$, $i = 1, 2$, for which $h = (h_1^+ - h_1^-) + i(h_2^+ - h_2^-)$, share the property assumed to hold for h . Therefore, it suffices to prove the result in the case where h is positive. Since h is ν -essentially bounded and ν -measurable, the ν -measurable function fh is ν -integrable for all $f \in L^1(\Omega, \nu)$. Let $\mathcal{C}_c(\Omega)$ denote the space of compactly supported continuous complex functions on Ω . We notice that the positive functional $I_h : \mathcal{C}_c(\Omega) \rightarrow \mathbb{C}$, defined by $I_h(f) = \int f h d\nu$, satisfies the relation

$$I_h(f) = \int f(x) h(x) d\nu(x) = \int f(yx) h(yx) d\nu(x) = \int f(yx) h(x) d\nu(x) = I_h(f(y \cdot))$$

for each $y \in \Omega$, and hence is a left Haar integral in the group Ω . By the uniqueness theorem of Haar integrals, there is a $c > 0$, such that $I_h(f) = c \int f d\nu$ for all $f \in \mathcal{C}_c(\Omega)$. Since $\mathcal{C}_c(\Omega)$ is dense in $L^1(\Omega, \nu)$, it follows that $h(x) = c$ for almost all $x \in \Omega$.

(b) Fix some $A \in \mathcal{B}(\Omega)$, such that $0 < \nu(A) < \infty$, and denote $s = \nu(A)^{-1} \int_A h d\nu \in X$. Let $w^* \in X^*$. Since h is ν -measurable, so is the complex valued function $x \mapsto \langle w^*, h(x) \rangle$, which thus coincides almost everywhere with some Borel function h_{w^*} . Since $(x, y) \mapsto xy$ is continuous, the function $(x, y) \mapsto h_{w^*}(xy)$ is $\nu \times \nu$ -measurable. By assumption, h_{w^*} satisfies the conditions of (a), so there is a constant $c \in \mathbb{C}$, and a ν -null set N , such that $h_{w^*}(y) = c$ for all $y \in \Omega \setminus N$. Let $x \in \Omega$. Since the left and right Haar measures have the same null sets, also $Nx^{-1} \cup x^{-1}N$ is a ν -null set. Thus, for each $x \in \Omega$, we have $h_{w^*}(yx) = c = h_{w^*}(xy)$ for almost all $y \in \Omega$.

Using this fact, the assumption and the Fubini-Tonelli theorem, we get for each $f \in L^1(\Omega, \nu)$,

$$\begin{aligned}
\langle w^*, \nu(A)s \int f d\nu \rangle &= \int_A h_{w^*}(x) d\nu(x) \int f(y) d\nu(y) \\
&= \int \left(\int \chi_A(x) h_{w^*}(x) f(y) d\nu(x) \right) d\nu(y) \\
&= \int \left(\int \chi_A(x) h_{w^*}(yx) f(y) d\nu(x) \right) d\nu(y) \\
&= \int \left(\int \chi_A(x) h_{w^*}(yx) f(y) d\nu(y) \right) d\nu(x) \\
&= \int \left(\int \chi_A(x) h_{w^*}(xy) f(y) d\nu(y) \right) d\nu(x) \\
&= \int \left(\int \chi_A(x) h_{w^*}(y) f(y) d\nu(y) \right) d\nu(x) = \langle w^*, \nu(A) \int f(y) h(y) d\nu(y) \rangle.
\end{aligned}$$

The use of the Fubini-Tonelli theorem is justified because ν is σ -finite, $(x, y) \mapsto \chi_A(x) h_{w^*}(yx) f(y)$ is $\nu \times \nu$ -measurable, and

$$\int \left(\int \|\chi_A(x) h_{w^*}(yx) f(y)\| d\nu(x) \right) d\nu(y) \leq \nu(A) \|w^*\| M \|f\|_1 < \infty,$$

where $M > 0$ is such that $\|h(x)\| \leq M$ for almost all $x \in \Omega$. Since $w^* \in X^*$ was arbitrary, we get $\int_B h(y) d\nu(y) = \int_B s d\nu(y)$ for each $B \in \mathcal{B}(\Omega)$ of finite measure. Thus, from [6, Corollary 5, p. 47] and the σ -finiteness of ν it follows that $h(x) = s$ for almost all $x \in \Omega$. \square

Proposition 1. *Assume that Ω is a locally compact second countable topological group with a left Haar measure ν , and X a Banach space having the Radon-Nikodým property. In addition, assume that there is a homomorphism $\alpha : \Omega \rightarrow \text{Iso}(X)$, such that*

- (i) $\sup_{x \in \Omega} \|\alpha(x)\| < \infty$;
- (ii) *for all $w \in X$, the map $x \mapsto \alpha(x^{-1})(w)$ is ν -measurable.*

If $\Gamma : L^1(\Omega, \nu) \rightarrow X$ is a continuous linear map satisfying $\alpha(x)(\Gamma(f)) = \Gamma(f(x^{-1}\cdot))$ for all $f \in L^1(\Omega, \nu)$ and $x \in \Omega$, then there is a unique vector $s \in X$, such that

$$\Gamma(f) = \int f(x) \alpha(x)(s) d\nu(x)$$

for all $f \in L^1(\Omega, \nu)$. If each $\alpha(x)$ is an isometry, then $\|s\| = \|\Gamma\|$.

Proof. Let $v : \Omega \rightarrow X$ be the function obtained in Lemma 3. We have to prove that for some unique $s \in X$, it satisfies $v(x) = \alpha(x)(s)$ for almost all $x \in \Omega$. To that end, let $B \in \mathcal{B}(\Omega)$ be such that $\nu(B) < \infty$, and $y \in \Omega$. Then, by the continuity of the linear map $\alpha(y)$ we get, by using the left invariance of ν ,

$$\begin{aligned}
\int_B \alpha(y)(v(x)) d\nu(x) &= \alpha(y)(\Gamma(\chi_B)) = \Gamma(\chi_B(y^{-1}\cdot)) = \int \chi_B(y^{-1}x) v(x) d\nu(x) \\
&= \int_B v(yx) d\nu(x).
\end{aligned}$$

Since the measure ν is σ -finite, it follows from [6, Corollary 5, p. 47] that for each $y \in \Omega$,

$$(4) \quad \alpha(y)(v(x)) = v(yx) \text{ for almost all } x \in \Omega.$$

Now define the map $v_0 : \Omega \rightarrow X$ by $v_0(x) = \alpha(x^{-1})(v(x))$. Then v_0 is ν -measurable. Indeed, let $B \in \mathcal{B}(\Omega)$ be such that $\nu(B) < \infty$. Since v is ν -measurable, there is a sequence (v_n) of ν -simple functions vanishing outside B and converging ν -a.e. to $\chi_B v$. For each $w \in X$, the map $x \mapsto \alpha(x^{-1})(w)$ is ν -measurable by assumption (ii), so that also the functions $x \mapsto \alpha(x^{-1})(v_n(x))$, are ν -measurable. Now $\alpha(x^{-1})(v_n(x)) \rightarrow \chi_B(x)\alpha(x^{-1})(v(x)) = \chi_B v_0(x)$ for ν -almost all x , because $\alpha(x^{-1})$ is continuous, so the limit $\chi_B v_0$ is ν -measurable [8, p. 150]. Thus v_0 is ν -measurable.

Let $f \in L^1(\Omega, \nu)$. Since v_0 is ν -measurable, so is $f v_0$ [8, p. 106]. In addition, since $\sup_{x \in \Omega} \|v(x)\| = \|\Gamma\|$, we get $\|f(x)v_0(x)\| \leq M|f(x)|\|v(x)\| \leq |f(x)|M\|\Gamma\|$ for all x , where $M = \sup_{x \in \Omega} \|\alpha(x)\| < \infty$, so $f v_0$ is ν -integrable. In particular, v_0 is integrable over any set $B \in \mathcal{B}(\Omega)$ of finite measure. Also, $\|v_0(x)\| \leq M\|\Gamma\|$ for all x , so v_0 is ν -essentially bounded. Since α is a homomorphism, $v(x) = \alpha(x)(v_0(x))$ for all x . Let $y \in \Omega$. The result (4) gives $\alpha(y)(\alpha(x)(v_0(x))) = \alpha(yx)(v_0(yx))$ for almost all x , so that

$$(5) \quad \text{for each } y \in \Omega, \quad v_0(x) = v_0(yx) \text{ for almost all } x \in \Omega.$$

By Lemma 4 there is an $s \in X$, such that $v_0(x) = s$ for ν -almost all x . Thus

$$\Gamma(f) = \int f(x)v(x)d\nu(x) = \int f(x)\alpha(x)(v_0(x))d\nu(x) = \int f(x)\alpha(x)(s)d\nu(x)$$

for all $f \in L^1(\Omega, \nu)$. The vector s in the above representation is uniquely determined, because if $s' \in X$ had the same properties, then by the uniqueness of the map $x \mapsto v(x)$ in the representation of Lemma 3, $\alpha(x)(s) = v(x) = \alpha(x)(s')$ for almost all $x \in \Omega$, so that $s = s'$.

If $\alpha(x)$ is an isometry for each $x \in \Omega$, we have in addition,

$$\|\Gamma\| = \sup_{x \in \Omega} \|v(x)\| = \sup_{x \in \Omega} \|\alpha(x)(s)\| = \|s\|$$

The proof is complete. \square

4. POSITIVE NORMAL COVARIANT MAPS

Now we return to the concept of (G, β, d) introduced earlier. Theorem 2 below characterizes all positive normal β -covariant maps $\Gamma : L^\infty(G, \lambda) \rightarrow L(\mathcal{H})$. The proof is based on the fact that $\mathcal{T}(\mathcal{H})$, being a separable dual space, has the Radon-Nikodým property by [6, p. 79]. Therefore, the following Lemma is needed. We give the proof for completeness. (The result is given without proof e.g. in [14, Exercise 5.7, p. 131].)

Lemma 5. *The space $\mathcal{T}(\mathcal{H})$ is separable (with respect to the trace norm).*

Proof. If $\varphi, \psi \in \mathcal{H}$ are such that $\|\varphi\| = 1$, and $\|\varphi - \psi\| \leq 1$, then

$$(6) \quad \| |\psi\rangle\langle\psi| - |\varphi\rangle\langle\varphi| \|_{\text{tr}} \leq 3\|\psi - \varphi\|.$$

Indeed, since the map $\mathcal{T}(\mathcal{H}) \ni T \mapsto \text{Tr}[T \cdot] \in \mathcal{C}(\mathcal{H})^*$, where $\mathcal{C}(\mathcal{H})$ denotes the set of compact operators, is an isometry, we have $\|T\|_{\text{tr}} = \sup\{|\text{Tr}[TA]| \mid A \in \mathcal{C}(\mathcal{H}), \|A\| \leq 1\}$ for each

$T \in \mathcal{T}(\mathcal{H})$. Let $\varphi, \psi \in \mathcal{H}$ be such that $\|\varphi\| = 1$, and $\|\varphi - \psi\| \leq 1$. If $A \in \mathcal{C}(\mathcal{H})$, $\|A\| \leq 1$, we have

$$\begin{aligned} |\operatorname{Tr}[(|\psi\rangle\langle\psi| - |\varphi\rangle\langle\varphi|)A]| &= |\langle\psi|A\psi\rangle - \langle\varphi|A\varphi\rangle| \leq |\langle\psi|A\psi\rangle - \langle\psi|A\varphi\rangle| + |\langle\psi|A\varphi\rangle - \langle\varphi|A\varphi\rangle| \\ &\leq \|\psi\|\|\psi - \varphi\| + \|\psi - \varphi\|\|\varphi\| \\ &\leq (\|\psi - \varphi\| + \|\varphi\|)\|\psi - \varphi\| + \|\psi - \varphi\|\|\varphi\| \leq 3\|\psi - \varphi\|. \end{aligned}$$

Thus (6) holds.

Let M be a countable dense set in the separable space \mathcal{H} . Define \mathcal{F} to be the set of operators of the form $\sum_{\psi \in F} \lambda_\psi |\psi\rangle\langle\psi|$, where F is a finite subset of M and each λ_ψ is a positive rational number (the vectors ψ need not be of unit length). Since M and \mathbb{Q} are countable sets, \mathcal{F} is countable. Clearly \mathcal{F} is a subset of the set $\mathcal{T}(\mathcal{H})^+$ of positive trace-class operators. We proceed to show that \mathcal{F} is $\|\cdot\|_{\operatorname{tr}}$ -dense in $\mathcal{T}(\mathcal{H})^+$.

Let $S \in \mathcal{T}(\mathcal{H})^+$ and $\epsilon > 0$. Using the decomposition $S = \sum_n t_n |\varphi_n\rangle\langle\varphi_n|$, which converges in the trace norm, with $t_n \geq 0$ and the φ_n orthonormal unit vectors, we find that there is a $k \in \mathbb{N}$, such that

$$(7) \quad \left\| S - \sum_{n=1}^k t_n |\varphi_n\rangle\langle\varphi_n| \right\|_{\operatorname{tr}} < \frac{\epsilon}{3}.$$

Now we choose positive rational numbers λ_n , $n = 1, \dots, k$, such that $|t_n - \lambda_n| < \frac{\epsilon}{3k}$ for all $n = 1, \dots, k$. Then

$$(8) \quad \left\| \sum_{n=1}^k t_n |\varphi_n\rangle\langle\varphi_n| - \sum_{n=1}^k \lambda_n |\varphi_n\rangle\langle\varphi_n| \right\|_{\operatorname{tr}} < \frac{\epsilon}{3}.$$

Since M is dense, we can pick vectors $\psi_n \in M$, $n = 1, \dots, k$, such that $\|\psi_n - \varphi_n\| < \frac{\epsilon}{9 \sum_{n=1}^k \lambda_n}$ for all $n = 1, \dots, k$. It can be assumed that $\epsilon < 1$, so that we can use the result (6) to get

$$(9) \quad \left\| \sum_{n=1}^k \lambda_n |\varphi_n\rangle\langle\varphi_n| - \tilde{S} \right\|_{\operatorname{tr}} < \frac{\epsilon}{3},$$

where $\tilde{S} = \sum_{n=1}^k \lambda_n |\psi_n\rangle\langle\psi_n| \in \mathcal{F}$. The inequalities (7)-(9) now imply $\|S - \tilde{S}\| < \epsilon$. Thus \mathcal{F} is $\|\cdot\|_{\operatorname{tr}}$ -dense in $\mathcal{T}(\mathcal{H})^+$.

Since $\mathcal{T}(\mathcal{H}) = \mathcal{T}(\mathcal{H})^+ - \mathcal{T}(\mathcal{H})^+ + i(\mathcal{T}(\mathcal{H})^+ - \mathcal{T}(\mathcal{H})^-)$, the set $\mathcal{F} - \mathcal{F} + i(\mathcal{F} - \mathcal{F})$ is a countable dense subset of $\mathcal{T}(\mathcal{H})$. \square

Theorem 2. Let $\Gamma : L^\infty(G, \lambda) \rightarrow L(\mathcal{H})$ be a normal positive β -covariant linear map satisfying $\Gamma(g \mapsto 1) = I$. Then Γ is of the form of Theorem 1 for a unique positive operator $T \in \mathcal{T}(\mathcal{H})$ of trace one.

Proof. Since $\Gamma : L^1(G, \lambda)^* \rightarrow \mathcal{T}(\mathcal{H})^*$ is a weak-* continuous linear map, there is a linear map $\Gamma_* : \mathcal{T}(\mathcal{H}) \rightarrow L^1(G, \lambda)$, such that $(\Gamma_*)^* = \Gamma$. The map Γ_* is also positive, since $\int (\Gamma_*(S))(g) f(g) d\lambda(g) = \operatorname{Tr}[\Gamma(f)S] \geq 0$ for all positive $S \in \mathcal{T}(\mathcal{H})$ and $f \in L^\infty(G, \lambda)$, $f \geq 0$. Let $S \in \mathcal{T}(\mathcal{H})$ be positive and $f \in L^\infty(G, \lambda) \cap L^1(G, \lambda)$ a positive function. Then $\Gamma(f)$ is a positive operator and $\Gamma_*(S)$ a positive function. By covariance, we have

$$\operatorname{Tr}[\Gamma(f)\beta(g)(S)] = \operatorname{Tr}[\beta(g)^*(\Gamma(f))S] = \operatorname{Tr}[S\Gamma(f(g \cdot))] = \int (\Gamma_*(S))(g') f(gg') d\lambda(g'),$$

from which it follows by the Fubini-Tonelli theorem and the right invariance of λ that

$$\int \text{Tr}[\Gamma(f)\beta(g)(S)]d\lambda(g) = \int (\Gamma_*(S))(g') \left(\int f(gg')d\lambda(g) \right) d\lambda(g') = \|\Gamma_*(S)\|_1 \|f\|_1 < \infty.$$

Lemma 2 now implies that $\Gamma(f) \in \mathcal{T}(\mathcal{H})$, and since $\|\Gamma_*(S)\|_1 = \int \Gamma_*(S)(g)d\lambda(g) = \text{Tr}[S\Gamma(g \mapsto 1)] = \text{Tr}[S]$, we find (by using Lemma 2 again) that for positive $f \in L^\infty(G, \lambda) \cap L^1(G, \lambda)$, $\|\Gamma(f)\|_{\text{tr}} = d^{-1}\|f\|_1$. If $f \in L^1(G, \lambda) \cap L^\infty(G, \lambda)$ is arbitrary, we can write $f = (f_1^+ - f_1^-) + i(f_2^+ - f_2^-)$, where the f_i^\pm are positive, and $f_1^+ + f_1^- + f_2^+ + f_2^- = |f_1| + |f_2| \leq 2\|f\|_1$. It then follows by the linearity of Γ that $\|\Gamma(f)\|_{\text{tr}} \leq 2d^{-1}\|f\|_1$, implying that the restriction $\Gamma|_{L^1(G, \lambda) \cap L^\infty(G, \lambda)} : L^1(G, \lambda) \cap L^\infty(G, \lambda) \rightarrow \mathcal{T}(\mathcal{H})$ is continuous with respect to the norms $\|\cdot\|_1$ and $\|\cdot\|_{\text{tr}}$. Since the set $L^1(G, \lambda) \cap L^\infty(G, \lambda)$ contains all integrable simple functions, it is dense in $L^1(G, \lambda)$. Therefore (since $\mathcal{T}(\mathcal{H})$ is complete), $\Gamma|_{L^1(G, \lambda) \cap L^\infty(G, \lambda)}$ can be extended to a continuous linear map $\tilde{\Gamma} : L^1(G, \lambda) \rightarrow \mathcal{T}(\mathcal{H})$.

The map $\tilde{\Gamma}$ is positive. In fact, if $f \in L^1(G, \lambda)$ is positive, there is an increasing sequence (f_n) of integrable positive simple functions converging pointwise to f . By the monotone convergence theorem, $f_n \rightarrow f$ in the $\|\cdot\|_1$ -norm, so that the trace-class operator $\tilde{\Gamma}(f)$, being the trace-norm (and hence weak) limit of the sequence $\Gamma(f_n)$ of positive trace-class operators, must be positive.

Now we show that the conditions of Proposition 1 are satisfied by the measure space $(G, \mathcal{B}(G), \lambda)$, the Banach space $\mathcal{T}(\mathcal{H})$, the homomorphism β , and the linear map $\tilde{\Gamma}$.

Since $\mathcal{T}(\mathcal{H}) \cong \mathcal{C}(\mathcal{H})^*$ is separable by Lemma 5, it has the Radon-Nikodým property [6, p. 79]. Since each $\beta(g)$ is an isometry, the condition (i) holds. Let $S \in \mathcal{T}(\mathcal{H})$ and $A \in L(\mathcal{H})$. Since λ is a Borel measure, the map $g \mapsto \text{Tr}[A\beta(g^{-1})(S)]$, being continuous, is also λ -measurable. Thus $G \ni g \mapsto w^*(\beta(g^{-1})(S)) \in \mathbb{C}$ is λ -measurable for each $w^* \in \mathcal{T}(\mathcal{H})^* \cong L(\mathcal{H})$. Since $\mathcal{T}(\mathcal{H})$ is separable, this implies by [8, p. 149] that the map $g \mapsto \beta(g^{-1})(S)$ is measurable, so that the condition (ii) of Proposition 1 is satisfied. To verify condition (iii), let $f \in L^1(G, \lambda)$, $g \in G$. Choose a sequence (f_n) of integrable simple functions converging to f in the $\|\cdot\|_1$ -norm. Thus, by the continuity of the mappings involved, the covariance of Γ , and the fact that the map $\beta(g^{-1})^* = (\beta(g)^{-1})^*$ coincides with $\beta(g)$ on $\mathcal{T}(\mathcal{H})$, we get

$$\beta(g)(\tilde{\Gamma}(f)) = \lim_n \beta(g^{-1})^*(\Gamma(f_n)) = \lim_n \Gamma(f_n(g^{-1}\cdot)) = \tilde{\Gamma}(f(g^{-1}\cdot)),$$

where the limits are in the trace norm and the $\|\cdot\|_1$ -norm. This proves that (iii) holds.

Thus, we can apply Proposition 1 to the map $\tilde{\Gamma}$. There is a unique $T' \in \mathcal{T}(\mathcal{H})$, such that

$$\tilde{\Gamma}(f) = \int f(g)\beta(g)(T')d\lambda(g)$$

for all $f \in L^1(G, \lambda)$. Since $L^\infty(G, \lambda) \cap L^1(G, \lambda)$ is weak-* dense in $L^\infty(G, \lambda)$ and Γ is normal, we also have

$$\Gamma(f) = \int f(g)\beta(g)(T')d\lambda(g)$$

in the ultraweak sense for all $f \in L^\infty(G, \lambda)$.

It remains to prove that T' is positive and of trace d^{-1} .

Let $S \in \mathcal{T}(\mathcal{H})$ be positive. Since $\Gamma(\chi_B)$ is a positive operator, we have

$$0 \leq \text{Tr}[S\Gamma(\chi_B)] = \int_B \text{Tr}[S\beta(g)(T')]d\lambda(g)$$

for all $B \in \mathcal{B}(G)$, from which it follows by the continuity of $g \mapsto \text{Tr}[S\beta(g)(T')]$ that $\text{Tr}[S\beta(g)(T')] \geq 0$ for all $g \in G$. Thus T' must be positive.

In addition, by the condition $\Gamma(g \mapsto 1) = I$, and Lemma 2,

$$\text{Tr}[S]d^{-1} = d^{-1}\text{Tr}[S\Gamma(\chi_G)] = d^{-1} \int \text{Tr}[S\beta(g)(T')]d\lambda(g) = \text{Tr}[S]\text{Tr}[T']$$

for any positive $S \in \mathcal{T}(\mathcal{H})$. Thus $\text{Tr}[T'] = d^{-1}$, so that by defining $T = T'd$, we get the required form for Γ . \square

5. COVARIANT OBSERVABLES

An observable (i.e. a positive normalized operator measure) $E : \mathcal{B}(G) \rightarrow L(\mathcal{H})$ is said to be β -covariant if $\beta(g)^*(E(B)) = E(g^{-1}B)$ for all $g \in G$ and $B \in \mathcal{B}(G)$. The following Lemma shows that Theorem 2 can be used to characterize the covariant observables. The result (b) of the Lemma is obtained in [9] for the more general case where the group need not be unimodular, and the condition (2) is not assumed. In the context of this paper, the proof following [10] is more simple, as it can be formulated so that it uses Lemma 2. The proof is therefore given here.

Lemma 6. *Let $E : \mathcal{B}(G) \rightarrow L(\mathcal{H})$ be an observable.*

- (a) *Assume that for each trace class operator S , the measure $B \mapsto \text{Tr}[SE(B)]$ is continuous with respect to the measure λ . Then for each $f \in L^\infty(G, \lambda)$, the operator integral $\int fdE$ exists in $L(\mathcal{H})$ in the ultraweak sense, and the linear map $f \mapsto \int fdE$ is normal, positive, and satisfies $\int 1dE(g) = I$. If E is β -covariant, so is the map $f \mapsto \int fdE$.*
- (b) *If E is β -covariant, the measure $B \mapsto \text{Tr}[SE(B)]$ is continuous with respect to the measure λ for each trace class operator S .*

Proof. (a) Let $S \in \mathcal{T}(\mathcal{H})$. Then $S = \sum_n t_n |\psi_n\rangle\langle\varphi_n|$, where (φ_n) and (ψ_n) are orthonormal sequences, $t_n \geq 0$, and $\sum t_n < \infty$. The series converges in the trace norm. The map μ , defined by $B \mapsto \mu(B) = \text{Tr}[SE(B)]$ is a complex valued finite measure, and (by the $\|\cdot\|_{\text{tr}}$ -continuity of the trace functional) it is a pointwise limit of the measures $\sum_{n=1}^k \mu_n$, where $\mu_n(B) = t_n \text{Tr}[|\psi_n\rangle\langle\varphi_n|E(B)]$ for each $B \in \mathcal{B}(G)$. Since the total variation norm of μ_n satisfies $\|\mu_n\| \leq 4 \sup_{B \in \mathcal{B}(G)} |\mu_n(B)| \leq 4t_n$, the series $\mu = \sum_n \mu_n$ converges absolutely in the total variation norm.

Let $f \in L^\infty(G, \lambda)$. Since μ and each μ_n are λ -continuous, $|f(g)| \leq \|f\|_\infty$ also μ_n -, and μ -almost everywhere. Thus, $\int |f|d\mu_n \leq \|f\|_\infty \|\mu_n\| \leq 4\|f\|_\infty t_n$ so that $\sum_n \int |f|d\mu_n \leq 4\|f\|_\infty \sum_n t_n = 4\|f\|_\infty \|S\|_{\text{tr}} < \infty$. It now follows e.g. from [11, Lemma 1] that f is μ -integrable, and

$$\int fd(\text{Tr}[SE(\cdot)]) = \int fd\mu = \sum_n \int fd\mu_n = \sum_n t_n \int fd(\text{Tr}[|\psi_n\rangle\langle\varphi_n|E(\cdot)]).$$

Since μ is λ -continuous, the integral does not depend on the representative of $f \in L^\infty(G, \lambda)$. In addition,

$$(10) \quad \left| \int fd(\text{Tr}[SE(\cdot)]) \right| \leq \sum_n \int |f|d\mu_n = 4\|f\|_\infty \|S\|_{\text{tr}},$$

so that the functional $S \mapsto \int f d(\text{Tr}[SE(\cdot)])$ is $\|\cdot\|_{\text{tr}}$ -continuous. Thus the integral $\int f dE$ exists in the ultraweak sense as an operator in $L(\mathcal{H})$, i.e., for each $S \in \mathcal{T}(\mathcal{H})$,

$$(11) \quad \text{Tr}[S(\int f dE)] = \int f d(\text{Tr}[SE(\cdot)]).$$

Since $B \mapsto \text{Tr}[SE(B)]$ is λ -continuous, it has a density $g_S \in L^1(G, \lambda)$. Since $L^\infty(G, \lambda) \ni f \mapsto \int f dE \in L(\mathcal{H})$ is the dual map of $\mathcal{T}(\mathcal{H}) \ni S \mapsto g_S \in L^1(G, \lambda)$, it is normal.

Let $f \in L^\infty(G, \lambda)$ be positive and $S \in \mathcal{T}(\mathcal{H})$ a positive operator. Since the measure $\text{Tr}[SE(\cdot)]$ is positive, so is $\text{Tr}[S(\int f dE)] = \int f d(\text{Tr}[SE(\cdot)])$. It follows that $\int f dE$ is positive. Thus the map $f \mapsto \int f dE$ is positive. Since E is normalized, $\int 1 dE(g) = E(G) = I$.

Assume now that E is β -covariant. Let $g \in G$, $B \in \mathcal{B}(G)$, and $S \in \mathcal{T}(\mathcal{H})$. Since the measure $\text{Tr}[SE(\cdot)]$ has the density $g_S \in L^1(G, \lambda)$, the measure $\text{Tr}[SE(g^{-1}\cdot)]$ has the density $g_S(g^{-1}\cdot)$. Using the left invariance of λ and the covariance of E , we get

$$\begin{aligned} \text{Tr}[S\beta(g)^*(\int f dE)] &= \text{Tr}[\beta(g)(S)(\int f dE)] = \int f d(\text{Tr}[\beta(g)(S)E(\cdot)]) = \int f d(\text{Tr}[SE(g^{-1}\cdot)]) \\ &= \int f(g')g_S(g^{-1}g')d\lambda(g') = \int f(gg')g_S(g')d\lambda(g') = \int f(g\cdot)d(\text{Tr}[SE(\cdot)]) \\ &= \text{Tr}[S(\int f(g\cdot)dE)], \end{aligned}$$

which proves that the map $f \mapsto \int f dE$ is β -covariant.

(b) Let $S \in \mathcal{T}(\mathcal{H})$ be positive and of trace one, and μ the probability measure $B \mapsto \text{Tr}[SE(B)]$. Now for each $B \in \mathcal{B}(G)$, covariance implies

$$\text{Tr}[\beta(g)^*(E(B))S] = \text{Tr}[SE(g^{-1}B)] = \int \chi_{g^{-1}B} d\mu = \int \chi_B(gg') d\mu(g').$$

Thus, by Lemma 2, the Fubini-Tonelli theorem, and the right invariance of λ , we get

$$\begin{aligned} \text{Tr}[E(B)] &= d^{-1} \int \text{Tr}[E(B)\beta(g)(S)] d\lambda(g) = d^{-1} \int \left(\int \chi_B(gg') d\lambda(g) \right) d\mu(g') \\ &= d^{-1} \lambda(B) \int d\mu = d^{-1} \lambda(B). \end{aligned}$$

Now let $S \in \mathcal{T}(\mathcal{H})$ be arbitrary. Then, if $B \in \mathcal{B}(G)$ is such that $\lambda(B) < \infty$, we have $|\text{Tr}[SE(B)]| \leq \|S\| \|E(B)\|_{\text{tr}} = d^{-1} \|S\| \lambda(B)$. This implies that the measure $B \mapsto \text{Tr}[SE(B)]$ is λ -continuous. \square

Theorem 3. *Let $E : \mathcal{B}(G) \rightarrow L(\mathcal{H})$ be a positive normalized β -covariant operator measure. Then*

$$E(B) = d^{-1} \int_B \beta(g)(T) d\lambda(g)$$

in the ultraweak sense, for some unique positive operator $T \in \mathcal{T}(\mathcal{H})$ of trace one.

Proof. By the previous Lemma, the linear map $L^\infty(G, \lambda) \ni f \mapsto \int f dE \in L(\mathcal{H})$ satisfies the conditions of Theorem 2 and hence is of the form

$$\int f dE = d^{-1} \int f(g) \beta(g)(T) d\lambda(g)$$

for some unique positive operator T of trace one. In particular,

$$(12) \quad E(B) = \int \chi_B dE = d^{-1} \int_B \beta(g)(T) d\lambda(g)$$

for each $B \in \mathcal{B}(G)$. The operator T in the representation (12) of E is also uniquely determined. In fact, if $S \in \mathcal{T}(\mathcal{H})$ is such that $E(B) = d^{-1} \int_B \beta(g)(S) d\lambda(g)$ for each $B \in \mathcal{B}(G)$, then by the uniqueness of T in the representation of the linear map $f \mapsto \int f dE$, we get $\int \chi_B(g) \beta(g)(S) d\lambda(g) = \int \chi_B(g) \beta(g)(T) d\lambda(g)$ for all $B \in \mathcal{B}(G)$, so $\beta(g)(S) = \beta(g)(T)$ for almost all g , showing that $S = T$. \square

Remark. Consider the concrete case $(\mathbb{R}^{2n}, \gamma, (2\pi)^n)$. For a linear map $\Gamma : L^\infty(\mathbb{R}^{2n}, \mu_L) \rightarrow L(L^2(\mathbb{R}^n))$, covariance means that $\gamma(x)(\Gamma(f)) = f(\cdot - x)$ for all $x \in \mathbb{R}^{2n}$ and $f \in L^\infty(\mathbb{R}^{2n}, \mu_L)$, whereas a covariant observable $E : \mathcal{B}(\mathbb{R}^{2n}) \rightarrow L(L^2(\mathbb{R}^n))$ is such that $\gamma(x)(E(B)) = E(x + B)$ for each $x \in \mathbb{R}^{2n}$ and $B \in \mathcal{B}(\mathbb{R}^{2n})$. Thus Theorem 2 gives, in particular, a characterization of positive covariant linear maps $\Gamma : L^\infty(\mathbb{R}^{2n}, \mu_L) \rightarrow L(L^2(\mathbb{R}^n))$, and Theorem 3 a characterization of the covariant phase space observables.

6. A NOTE ON QUANTIZATION MAPS ON THE SET OF UNBOUNDED FUNCTIONS

Since many of the important dynamical variables in classical mechanics are unbounded functions, it is rather restrictive to consider only the quantization maps $\Gamma : L^\infty(G, \lambda) \rightarrow L(\mathcal{H})$.

Let $\mathcal{F}(G)$ denote the set of all complex Borel functions on G , and $\mathcal{O}(\mathcal{H})$ the set of all (not necessarily bounded) linear operators in \mathcal{H} . We call a map $\Gamma : \mathcal{F}(G) \rightarrow \mathcal{O}(\mathcal{H})$ linear if $\alpha\Gamma(f) + \beta\Gamma(h) \subset \Gamma(\alpha f + \beta h)$ for all $\alpha, \beta \in \mathbb{C}$ and $f, h \in \mathcal{F}(G)$. For each $f \in \mathcal{F}(G)$, we let $D(\Gamma(f))$ denote the domain of $\Gamma(f)$.

Let $E : \mathcal{B}(G) \rightarrow L(\mathcal{H})$ be a positive operator measure. For $f \in \mathcal{F}(G)$ let $D(f, E)$ be the set of those vectors $\varphi \in \mathcal{H}$ for which f is $E_{\psi, \varphi}$ -integrable for all $\psi \in \mathcal{H}$. The operator integral $L(f, E) = \int f dE$ is defined to be the unique (possibly unbounded) linear operator on the domain $D(f, E)$, for which $\langle \psi | L(f, E) \varphi \rangle = \int f dE_{\psi, \varphi}$ for all $\varphi \in D(f, E)$ and $\psi \in \mathcal{H}$ (cf. [12]). If f is real valued, then $L(f, E)$ is a symmetric operator.

Consider the map $\Gamma_E : \mathcal{F}(G) \rightarrow \mathcal{O}(\mathcal{H})$, defined by $\Gamma_E(f) = L(f, E)$. If $f, h \in \mathcal{F}(G)$, $\alpha, \beta \in \mathbb{C}$, then (since $|f + h| \leq |f| + |h|$) $\alpha\Gamma(f) + \beta\Gamma(h) \subset \Gamma(\alpha f + \beta h)$, so Γ_E is linear. It follows from the dominated convergence theorem that it is quasicontinuous in the sense of the following definition (already given in the Introduction).

Definition. A linear map $\Gamma : \mathcal{F}(G) \rightarrow \mathcal{O}(\mathcal{H})$ is *quasicontinuous*, if for each increasing sequence (f_n) of positive Borel functions converging pointwise to an $f \in \mathcal{F}(G)$ the numerical sequence $(\langle \psi | \Gamma(f_n) \varphi \rangle)$ converges to $\langle \psi | \Gamma(f) \varphi \rangle$ for all $\psi \in \mathcal{H}$ and $\varphi \in D(\Gamma(f)) \cap \bigcap_{n \in \mathbb{N}} D(\Gamma(f_n))$.

In the Introduction we mentioned that in order to be represented as an operator integral, a quantization map Γ must be at least positive, linear and quasicontinuous, and map bounded functions to $L(\mathcal{H})$, for then the map $E^\Gamma : \mathcal{B}(G) \rightarrow L(\mathcal{H})$, given by $B \mapsto \Gamma(\chi_B)$ is a positive operator measure, and $\Gamma(f) = L(f, E^\Gamma)$ for each bounded function $f \in \mathcal{F}(G)$. In order to claim that $\Gamma = L(\cdot, E^\Gamma)$, something must be assumed on the domains of the operators $\Gamma(f)$. The following simple result follows readily from the definition of the operator integral:

Proposition 2. *Let $\Gamma : \mathcal{F}(G) \rightarrow \mathcal{O}(\mathcal{H})$ be a linear map satisfying the following conditions:*

- (i) Γ is positive and quasicontinuous;
- (ii) Γ maps bounded functions to $L(\mathcal{H})$;
- (iii) for $f \in \mathcal{F}(G)$, the domain of $\Gamma(f)$ consists of those vectors $\varphi \in \mathcal{H}$ for which f is $E_{\psi, \varphi}^\Gamma$ -integrable for all $\psi \in \mathcal{H}$.

Then $\Gamma(f) = L(f, E^\Gamma)$ for all $f \in \mathcal{F}(G)$.

Proof. As (iii) asserts that the domains of the operators $\Gamma(f)$ and $L(f, E^\Gamma)$ are the same, we are left to show that $\Gamma(f)\varphi = L(f, E^\Gamma)\varphi$ for all φ in the common domain \mathcal{D} . Let $f \in \mathcal{F}(G)$, $\varphi \in \mathcal{D}$ and $\psi \in \mathcal{H}$. Assume first that f is positive. Pick an increasing sequence (f_n) of $\mathcal{B}(G)$ -simple functions converging pointwise to f . By (ii), $D(\Gamma(f_n)) = \mathcal{H}$, so quasicontinuity implies that the sequence (z_n^ψ) , where $z_n^\psi = \langle \psi | \Gamma(f_n) \varphi \rangle$, converges to $\langle \psi | \Gamma(f) \varphi \rangle$ for all $\psi \in \mathcal{H}$. Since each f_n is bounded, $\Gamma(f_n) = L(f_n, E^\Gamma)$ for all $n \in \mathbb{N}$, so $z_n^\psi = \int f_n dE_{\psi, \varphi}^\Gamma$. But now (iii) and the dominated convergence theorem imply that z_n^ψ converges to $\int f dE_{\psi, \varphi}^\Gamma = \langle \psi | L(f, E^\Gamma) \varphi \rangle$, so $\langle \psi | \Gamma(f) \varphi \rangle = \langle \psi | L(f, E^\Gamma) \varphi \rangle$. Since $\psi \in \mathcal{H}$ was arbitrary, this gives $\Gamma(f)\varphi = L(f, E^\Gamma)\varphi$. For a general $f \in \mathcal{F}(G)$, we write $f = f_1^+ - f_1^- + i(f_2^+ - f_2^-)$, where f_j^\pm are the positive and negative parts of f_j . Let $\varphi \in \mathcal{D}$ and $\psi \in \mathcal{H}$. Since $0 \leq f_j^\pm \leq |f|$, we have that also f_j^\pm is $E_{\psi, \varphi}^\Gamma$ -integrable for all $\psi \in \mathcal{H}$, i.e. $\varphi \in D(f_j^\pm, E^\Gamma) = D(\Gamma(f_j^\pm))$. Thus, $\Gamma(f_j^\pm)\varphi = L(f_j^\pm, E^\Gamma)\varphi$. By linearity, we get $\Gamma(f)\varphi = L(f, E^\Gamma)\varphi$, completing the proof. \square

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